ONLINE TIME-MODAL MONITORING OF NON STATIONNARY SYSTEMS

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ABSTRACT

A new algorithm for the online monitoring of varying modal parameters in vibrating structures subjected to unknown excitations is presented by using a vector autoregressive model. The method consists in applying a Short Time Sliding Window (STSW) on the signal. The solution, inside each sliding window, is found by applying a recursive multivariable least squares method via the computation of the QR factorization on the vector autoregressive model. In this method, only the R sub-matrices of the QR factorization need to be manipulated. An efficient model order is real time defined and updated. The minimum description length criterion is utilized to select an efficient model order which may be updated by increasing or decreasing order with respect to time from previous computational window. Various numerical and experimental data are presented to validate the proposed method; by investigating either abrupt or gradual changes in the system under white noise and various kinds of excitations such an impulsion, a sinusoidal or a random force. The results show that the modal parameters variation can be accurately identified and monitored but the monitoring of damping variation is more difficult if the system is continuously subjected to gradual changes or is excited by random excitations with high variances.

Keywords: Autoregressive model; recursive least squares; QR factorization updating; model order selection; modal parameter identification; varying system, sliding window.

1. VECTOR AUTOREGRESSIVE MODEL FOR MODAL ANALYSIS

The identification of structural modal parameters [1] plays an important role in structural health monitoring and is usually conducted by using experimental modal analysis methods in the frequency domain in a wide range of applications [2]. However, in several industrial applications [3] where it is not suitable to stop the machines or structures, the forces cannot be measured and are unknown. Since the forces result from natural excitations, operating modal analysis must fortunately be conducted [4, 5] for the monitoring of the structural modal parameters. Examples of such industrial applications can be found in bridge monitoring [6, 7], in identification of added mass and damping in fluid-structure interactions [8, 9], in crack detection [10] and in damage or crack monitoring of structures [11, 12]. The time domain has been found to be more suitable for operational modal analysis [13, 14] and several methods can be cited for the identification of time data, such as Ibrahim time domain method (ITD) [15], the least squares complex exponential (LSCE) [16], etc. Assuming a random environment, the excitation may be ignored and since modal analysis required multiple measurement locations, a vector autoregressive model should be applied [17] with a d sensor dimension and can be expressed as follows:

$$\{y(t)\}_{d\times 1} = [A]_{d\times dp}\{\varphi(t)\}_{dp\times 1} + \{e(t)\}_{d\times 1} \quad (1)$$

where $[A]_{d\times dp} = \begin{bmatrix} -[a_1] & -[a_2] & \dots & -[a_i] & \dots & -[a_p] \end{bmatrix}$ is the model parameter matrix,

$[a_i]_{d\times d}$ is the matrix of autoregressive parameters relating the output $\{y(t-i)\}$ to $\{y(t)\}$,

$\{\varphi(t)\}_{dp\times 1}$ is the regressor for the output vector $\{y(t)\}$,

$\{\varphi(t)\}^T = \begin{bmatrix} \{y(t-1)\}^T & \{y(t-2)\}^T & \dots & \{y(t-p)\}^T \end{bmatrix}$,

$\{y(t-i)\}_{d\times 1}$ (i=1,...,p) is the output vector with time delay $i\times T_s$,

T_s is the sampling period (s),

$\{e(t)\}_{d\times 1}$ is the residual vector of all output channels considered as the error of model.

When the data are assumed to be measured in a white noise environment, the least squares estimation may be assumed as unbiased [18]. If N successive output vectors of the responses from $\{y(t)\}$ to $\{y(t+N-1)\}$ are considered (N ≥ dp+d), the model parameters matrix $[A]_{d\times dp}$ can be expressed via the computation of the QR factorization [17] as follows:

$$[A] = ([R_{12}]^T[R_{11}]).([R_{11}]^T[R_{11}])^{-1} = ([R_{11}]^{-1}[R_{12}])^T \quad (2)$$

In this formula, $[R_{11}]$ and $[R_{12}]$ are sub-matrices of the upper triangular factor derived from the QR factorization of the data matrix as follows:

$$[K] = [Q][R] \quad (3)$$

where $[Q]_{N\times N}$ is an orthogonal matrix (that is $[Q][Q]^T = [I]$), $[R]$ has the form:

$$[R]_{N\times(dp+d)} = \begin{bmatrix} [R_{11}]_{dp\times dp} & [R_{12}]_{dp\times d} \\ 0 & [R_{22}]_{d\times d} \\ 0 & 0 \end{bmatrix} \quad (4)$$

and data matrix $[K]$ is constructed from N successive samples:

$$[K]_{N\times(dp+d)} = \begin{bmatrix} \{\varphi(t)\}^T & \{y(t)\}^T \\ \{\varphi(t+1)\}^T & \{y(t+1)\}^T \\ \dots & \dots \\ \{\varphi(t+N-1)\}^T & \{y(t+N-1)\}^T \end{bmatrix} \quad (5)$$

Once the model parameters matrix has been estimated, modal parameters can be directly identified from the eigendecomposition of the state matrix $[\Pi]$ [13].

$$[\Pi]_{dp\times dp}=\begin{bmatrix} -[a_1]_{d\times d} & -[a_2]_{d\times d} & -[a_i]_{d\times d} & \dots & -[a_p]_{d\times d} \\ [I] & 0 & 0 & \dots & 0 \\ 0 & [I] & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & [I] & 0 \end{bmatrix} \quad (6)$$

The requirement of selecting the model order is a disadvantage of autoregressive methods. A too low order will lead to erroneous results while a too large order will take too much computational time and may result in divergence. Consequently, it is suitable to find an efficient order that leads to a good compromise between precision and computational time [18-21]. An efficient order can be selected from various optimization-based criteria such as AIC [22] or MDL and other variants [23]. In this paper, we propose the using of MDL which is a very good criterion for short data modelling and consequently suitable for monitoring [23, 24].

$$MDL(p)=\frac{\log(\|\hat{e}(t)\|)}{d}+\log(1+\frac{2d.p}{N}\log N) \quad (7)$$

where $\|\hat{e}(t)\|$ is a norm of the estimated model error, which is taken from the sum of the main diagonal of the estimated error covariance matrix:

$$[\hat{E}]_{d\times d}=\frac{1}{N}[R_{22}]^T[R_{22}] \quad (8)$$

2. UPDATING METHODS

In structural health monitoring, the online survey of modal parameters requires an updating of algorithm. Since the model order can vary, the conventional recursive least squares updating algorithm [25] presents a certain amount of difficulties while changing the model order. In this paper, the parameter matrix is computed via the QR factorization and three methods are presented in order to update the solution with respect to both, time and model order where the efficient order is obtained directly from a previous time scheme.

2.1 Updating in time

The QR factorization at model order p should be recursively updated when a new set of samples data is available along with measuring time. From the matrices $[Q^{(k)}]$ and $[R^{(k)}]$ of data matrix $[K^{(k)}]$ at time $t = k$, one needs an update to $[Q^{(k+s)}]$ and $[R^{(k+s)}]$at time $t = k+s$ where the data matrix $[K^{(k+s)}]$ is found by deleting the first s rows and appending more s rows to matrix $[K^{(k)}]$.

$$[K^{(k)}]_{N\times(dp+d)} = \begin{bmatrix} \{\varphi(k)\}^T & \{y(k)\}^T \\ \{\varphi(k+1)\}^T & \{y(k+1)\}^T \\ \dots & \dots \\ \{\varphi(k+N-1)\}^T & \{y(k+N-1)\}^T \end{bmatrix} \quad (9)$$

$$[K^{(k+s)}]_{N\times(dp+d)} = \begin{bmatrix} \{\varphi(k+s)\}^T & \{y(k+s)\}^T \\ \{\varphi(k+s+1)\}^T & \{y(k+s+1)\}^T \\ \dots & \dots \\ \{\varphi(k+s+N-1)\}^T & \{y(k+s+N-1)\}^T \end{bmatrix} \quad (10)$$

The relationship can firstly be established as follows:

$$\begin{bmatrix} \multicolumn{2}{c}{[K^{(k)}]} \\ \{\varphi(k+1+N-1)\}^T & \{y(k+1+N-1)\}^T \\ \dots & \dots \\ \{\varphi(k+s+N-1)\}^T & \{y(k+s+N-1)\}^T \end{bmatrix}_{(N+s)\times(dp+d)} = \begin{bmatrix} \{\varphi(k)\}^T & \{y(k)\}^T \\ \dots & \dots \\ \{\varphi(k+s-1)\}^T & \{y(k+s-1)\}^T \\ \multicolumn{2}{c}{[K^{(k+s)}]} \end{bmatrix}_{(N+s)\times(dp+d)} \quad (11)$$

that gives in terms of the QR decomposition of the data matrix:

$$\begin{bmatrix} \multicolumn{2}{c}{[Q^{(k)}][R^{(k)}]} \\ \{\varphi(k+1+N-1)\}^T & \{y(k+1+N-1)\}^T \\ \dots & \dots \\ \{\varphi(k+s+N-1)\}^T & \{y(k+s+N-1)\}^T \end{bmatrix}_{(N+s)\times(dp+d)} = \begin{bmatrix} \{\varphi(k)\}^T & \{y(k)\}^T \\ \dots & \dots \\ \{\varphi(k+s-1)\}^T & \{y(k+s-1)\}^T \\ \multicolumn{2}{c}{[Q^{(k+s)}][R^{(k+s)}]} \end{bmatrix}_{(N+s)\times(dp+d)} \quad (12)$$

and in innovative form:

$$\begin{bmatrix} [Q^{(k)}] & 0 \\ 0 & [I_s] \end{bmatrix} \begin{bmatrix} \multicolumn{2}{c}{[R^{(k)}]} \\ \{\varphi(k+1+N-1)\}^T & \{y(k+1+N-1)\}^T \\ \dots & \dots \\ \{\varphi(k+s+N-1)\}^T & \{y(k+s+N-1)\}^T \end{bmatrix} = \begin{bmatrix} [I_s] & 0 \\ 0 & [Q^{(k+s)}] \end{bmatrix} \begin{bmatrix} \{\varphi(k)\}^T & \{y(k)\}^T \\ \dots & \dots \\ \{\varphi(k+s-1)\}^T & \{y(k+s-1)\}^T \\ \multicolumn{2}{c}{[R^{(k+s)}]} \end{bmatrix} \quad (13)$$

where $[I_s]_{s\times s}$ is the identity matrix.

In this algorithm, one wants an update of the sub-matrices $[R_{11}]$, $[R_{12}]$ and $[R_{22}]$ of matrix $[R]$ as defined in equation(4). Matrices $[R^{(k)}]$ and $[R^{(k+s)}]$ should therefore be partitioned in a well conditioned form:

$$\begin{bmatrix} [Q^{(k)}] & 0 \\ 0 & [I_s] \end{bmatrix} \begin{bmatrix} [R_1^{(k)}] & [R_2^{(k)}] \\ \{\varphi(k+1+N-1)\}^T & \{y(k+1+N-1)\}^T \\ \ldots & \ldots \\ \{\varphi(k+s+N-1)\}^T & \{y(k+s+N-1)\}^T \end{bmatrix} = \begin{bmatrix} [I_s] & 0 \\ 0 & [Q^{(k+s)}] \end{bmatrix} \begin{bmatrix} \{\varphi(k)\}^T & \{y(k)\}^T \\ \ldots & \ldots \\ \{\varphi(k+s-1)\}^T & \{y(k+s-1)\}^T \\ [R_1^{(k+s)}] & [R_2^{(k+s)}] \end{bmatrix} \quad (14)$$

where the new sub-matrices $[R_1^{(k)}]$ and $[R_2^{(k)}]$ are related to the sub-matrices $[R_{11}^{(k)}]$, $[R_{12}^{(k)}]$ and $[R_{22}^{(k)}]$ as follows:

$$[R_1^{(k)}]_{N\times dp} = \begin{bmatrix} [R_{11}^{(k)}]_{dp\times dp} \\ 0 \end{bmatrix} \text{ and } [R_2^{(k)}]_{N\times dp} = \begin{bmatrix} [R_{12}^{(k)}]_{dp\times d} \\ [R_{22}^{(k)}]_{(N-dp)\times d} \end{bmatrix} \quad (15)$$

If the first dp columns of equation (14) are extracted, we obtain:

$$\begin{bmatrix} [Q^{(k)}] & 0 \\ 0 & [I_s] \end{bmatrix} \begin{bmatrix} [R_1^{(k)}] \\ \{\varphi(k+1+N-1)\}^T \\ \ldots \\ \{\varphi(k+s+N-1)\}^T \end{bmatrix} = \begin{bmatrix} [I_s] & 0 \\ 0 & [Q^{(k+s)}] \end{bmatrix} \begin{bmatrix} \{\varphi(k)\}^T \\ \ldots \\ \{\varphi(k+s-1)\}^T \\ [R_1^{(k+s)}] \end{bmatrix} \quad (16)$$

It can be seen that equation (16) is a sub-problem of equation (12) for the first dp columns. The right hand side can then be transformed from the left one by using two sets of orthogonal Givens rotations, as described below.

The first set G_1 applies on the matrix $\begin{bmatrix} [R_1^{(k)}] \\ \{\varphi(k+1+N-1)\}^T \\ \ldots \\ \{\varphi(k+s+N-1)\}^T \end{bmatrix}$ to annihilate all $dp \times s$ elements on the last s rows (from 1st column to the last column and from low to up) to obtain an upper triangular matrix. It is seen that G_1 has the following form:

$$[G_1] = ([J_{N+1,dp}]\ldots[J_{N+s,dp}])\ldots([J_{N+1,1}]\ldots[J_{N+s,1}]) \quad (17)$$

where $[J_{i,j}]_{(N+s)\times(N+s)}$ is the Givens matrix zeroing the $(i,j)^{th}$ element of the matrix in the previous step $[J_{i+1,j}]\ldots[J_{N+s,j}]\ldots([J_{N+1,1}]\ldots[J_{N+s,1}]) \begin{bmatrix} [R_1^{(k)}] \\ \{\varphi(k+1+N-1)\}^T \\ \ldots \\ \{\varphi(k+s+N-1)\}^T \end{bmatrix}$ (let call $[S_{j+1,j}]$) and

$([J_{N+1,j}]...[J_{N+s,j}])$ is the set of matrices applied to the j^{th} column of the initial matrix $\begin{bmatrix} [R_1^{(k)}] \\ \{\varphi(k+1+N-1)\}^T \\ ... \\ \{\varphi(k+s+N-1)\}^T \end{bmatrix}$ from low to up.

These Givens matrices are easily constructed in common form as follows:

$$[J_{i,j}]_{(N+s)\times(N+s)} = \begin{bmatrix} 1 & ... & 0 & ... & 0 & ... & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & ... & \cos & ... & \sin & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & ... & -\sin & ... & \cos & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & ... & \underset{j}{0} & ... & \underset{i}{0} & ... & 1 \end{bmatrix} \begin{matrix} \\ \\ j \\ \\ i \\ \\ \\ \end{matrix} \tag{18}$$

where cos and sin are computed from two elements $S(j,j)$ and $S(i,j)$ as $\cos = \dfrac{S(j,j)}{\sqrt{S(j,j)^2 + S(i,j)^2}}$ and $\sin = \dfrac{S(i,j)}{\sqrt{S(j,j)^2 + S(i,j)^2}}$.

The left side term of (16) can be rewritten as:

$$\begin{bmatrix} [Q^{(k)}] & 0 \\ 0 & [I_s] \end{bmatrix} \begin{bmatrix} [R_1^{(k)}] \\ \{\varphi(k+1+N-1)\}^T \\ ... \\ \{\varphi(k+s+N-1)\}^T \end{bmatrix} = \begin{bmatrix} [Q^{(k)}] & 0 \\ 0 & [I_s] \end{bmatrix} [G_1^T][G_1] \begin{bmatrix} [R_1^{(k)}] \\ \{\varphi(k+1+N-1)\}^T \\ ... \\ \{\varphi(k+s+N-1)\}^T \end{bmatrix} = [\overline{Q}^{(k)}][G_1] \begin{bmatrix} [R_1^{(k)}] \\ \{\varphi(k+1+N-1)\}^T \\ ... \\ \{\varphi(k+s+N-1)\}^T \end{bmatrix} \tag{19}$$

The second set of Givens rotations $[G_2]$ is used to set unitary the first s rows and columns of the augmented matrix $[\overline{Q}^{(k)}] = \begin{bmatrix} [Q^{(k)}] & 0 \\ 0 & [I_s] \end{bmatrix} [G_1^T]$.

Consider vector $\{z_r^{(k)}\}_{(N+s)\times 1} = \{\overline{q}_r^{(k)}\}^T$ where $\{\overline{q}_r^{(k)}\}$ is the r^{th} row of the increased matrix $[\overline{Q}_{r-1}^{(k)}]$ at the $(r-1)^{th}$ computational step $(r = 1:s)$, since $\{z_r^{(k)}\}$ is orthonormal ($\{z_r^{(k)}\}^T \{z_r^{(k)}\} = 1$), it yields:

$$[G_{2,r}]\{z_r^{(k)}\} = [\overline{J}_{r+1}][\overline{J}_{r+2}]...[\overline{J}_{N+s}]\{z_r^{(k)}\} = \begin{bmatrix} 0 & ... & \underset{r}{1} & ... & 0 & ... & 0 \end{bmatrix}^T \tag{20}$$

where the Givens matrix $[\overline{J}_i]$ zeroing the i^{th} element of $\{z_r^{(k)}\}$ is in the form:

$$[\overline{J}_i]_{(N+s)\times(N+s)} = \begin{bmatrix} 1 & ... & 0 & 0 & ... & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & ... & \cos & \sin & ... & 0 \\ 0 & ... & -\sin & \cos & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & ... & \underset{(i-1)}{0} & \underset{(i)}{0} & ... & 1 \end{bmatrix} \begin{matrix} \\ \\ (i-1) \\ (i) \\ \\ \\ \end{matrix} \tag{21}$$

with $\cos = \dfrac{z_r^{(k)}(i-1)}{\sqrt{(z_r^{(k)}(i-1))^2 + (z_r^{(k)}(i))^2}}$ and $\sin = \dfrac{z_r^{(k)}(i)}{\sqrt{(z_r^{(k)}(i-1))^2 + (z_r^{(k)}(i))^2}}$.

Computing the matrix $[G_2]$ hence shows it to be equal to the multiplication of s components:

$$[G_2] = \prod_{r=s}^{1} [G_{2,r}] \tag{22}$$

The left side terms of (16) and (19) are thus further rewritten:

$$\begin{bmatrix} [Q^{(k)}] & 0 \\ 0 & [I_s] \end{bmatrix} \begin{bmatrix} [R_1^{(k)}] \\ \{\varphi(k+1+N-1)\}^T \\ ... \\ \{\varphi(k+s+N-1)\}^T \end{bmatrix} = \begin{bmatrix} [Q^{(k)}] & 0 \\ 0 & [I_s] \end{bmatrix} [G_1^T][G_2^T][G_2][G_1] \begin{bmatrix} [R_1^{(k)}] \\ \{\varphi(k+1+N-1)\}^T \\ ... \\ \{\varphi(k+s+N-1)\}^T \end{bmatrix} \tag{23}$$

Since the second Givens rotation set is established on the first s rows of the increased matrix, two interesting consequences are found:

- Its right transpose multiplication will unitary the first s rows and first s columns of augmented matrix $[\overline{Q}^{(k)}] = \begin{bmatrix} [Q^{(k)}] & 0 \\ 0 & 1 \end{bmatrix} [G_1^T]$;

- Its left multiplication will nonzero the first s elements of each row of the upper triangular matrix $[G_1]\begin{bmatrix}[R_1^{(k)}] \\ \{\varphi(k+N)\}^T\end{bmatrix}$, making each one an upper Hessenberg matrix.

That explains:

$$\begin{bmatrix}[Q^{(k)}] & 0 \\ 0 & [I_s]\end{bmatrix}[G_1^T][G_2^T] = \begin{bmatrix}[I_s] & 0 \\ 0 & [Q^{*(k+s)}]\end{bmatrix} \tag{24}$$

$$[G_2][G_1]\begin{bmatrix}[R_1^{(k)}] \\ \{\varphi(k+1+N-1)\}^T \\ \dots \\ \{\varphi(k+s+N-1)\}^T\end{bmatrix} = \begin{bmatrix}\{\varphi(k)\}^T \\ \dots \\ \{\varphi(k+s-1)\}^T \\ [R_1^{*(k+s)}]\end{bmatrix} \tag{25}$$

It can be seen that two Givens rotations sets are built only on the first dp columns of the data matrix. The derived matrix $[Q^{*(k+s)}]$ therefore coincides to the exact matrix $[Q^{(k+s)}]$ on the first dp columns and the orthonormal condition $\left[Q^{*(k+s)}\right]^T\left[Q^{*(k+s)}\right] = [I]$ is assured. Matrix $[R_1^{*(k+s)}]$ which is only nonzero on the first dp rows is the actual desired matrix $[R_1^{(k+s)}]$ and hence $[R_1^{*(k+s)}] = [R_1^{(k+s)}]$. Then the factorized matrices $[R_{11}^{(k+s)}]$ at the sample index $(k+s)$ are therefore exactly updated at this stage:

$$[R_{11}^{(k+s)}] = [[I]_{dp\times dp} \quad [0]_{dp\times(N-dp)}][R_1^{*(k+s)}] \tag{26}$$

With the derived matrix $Q^{*(k+s)}$, the last d columns of equations (14) can be rewritten as:

$$\begin{bmatrix}[Q^{(k)}] & 0 \\ 0 & [I_s]\end{bmatrix}\begin{bmatrix}[R_2^{(k)}] \\ \{y(k+1+N-1)\}^T \\ \dots \\ \{y(k+s+N-1)\}^T\end{bmatrix} = \begin{bmatrix}[I_s] & 0 \\ 0 & [Q^{*(k+s)}]\end{bmatrix}\begin{bmatrix}\{y(k)\}^T \\ \dots \\ \{y(k+s-1)\}^T \\ [R_2^{*(k+s)}]\end{bmatrix} \tag{27}$$

and the matrix $[R_2^{*(k+1)}]$ is directly extracted from the equality:

$$\begin{bmatrix}\{y(k)\}^T \\ \dots \\ \{y(k+s-1)\}^T \\ \{R_2^{*(k+s)}\}\end{bmatrix} = \begin{bmatrix}[I_s] & 0 \\ 0 & \left[Q^{*(k+s)}\right]^T\end{bmatrix}\begin{bmatrix}[Q^{(k)}] & 0 \\ 0 & [I_s]\end{bmatrix}\begin{bmatrix}[R_2^{(k)}] \\ \{y(k+1+N-1)\}^T \\ \dots \\ \{y(k+s+N-1)\}^T\end{bmatrix} \tag{28}$$

As discussed earlier, the first dp of matrix $[R_2^{*(k+s)}]$ are also exactly derived, which means the sub-matrix $[R_{12}^{(k+s)}]$ was exactly updated.

One can now write:

$$[R_2^{*(k+s)}] = \begin{bmatrix} [R_{12}^{(k+s)}]_{dp\times d} \\ [R_{22}^{*\,(k+s)}]_{(N-dp)\times d} \end{bmatrix} \tag{29}$$

Since $[Q^{(k+s)}]$ and $[Q^{*(k+s)}]$ are both orthogonal, we can readily see that the sub-matrix $[R_{22}^{*\,(k+s)}]$ satisfies the equation:

$$\left[R_{22}^{*\,(k+s)}\right]^T \left[R_{22}^{*\,(k+s)}\right] = \left[R_{22}^{(k+s)}\right]^T \left[R_{22}^{(k+s)}\right] \tag{30}$$

The error covariance matrix $[\hat{E}]$ (Eq. 8) is therefore updated.

2.2 Order updating

The data matrix $[K^{(p)}]$ at order p can be rewritten as:

$$[K^{(p)}]_{N\times(dp+d)} = \begin{bmatrix} \{\varphi(k)\}^T & \{y(k)\}^T \\ \{\varphi(k+1)\}^T & \{y(k+1)\}^T \\ \dots & \dots \\ \{\varphi(k+N-1)\}^T & \{y(k+N-1)\}^T \end{bmatrix} = \left[[K_1^{(p)}]_{N\times dp} \quad [K_2]_{N\times d}\right] \tag{31}$$

If the model order is updated to $p+1$, the data matrix has the form:

$$[K^{(p+1)}]_{N\times(d(p+1)+d)} = \left[[K_1^{(p)}]_{N\times dp} \quad [K^{'}]_{N\times d} \quad [K_2]_{N\times dp}\right] \tag{32}$$

where $[K^{'}]_{N\times d}$ are the added d columns:

$$[K^{'}] = \begin{bmatrix} \{y(k-(p+1))\}^T \\ \{y(k+1-(p+1))\}^T \\ \dots \\ \{y(k+N-1-(p+1))\}^T \end{bmatrix} \tag{33}$$

One can then compute the following matrix:

$$[Q^{(p)}]^T[K^{(p+1)}] = \left[[Q^{(p)}]^T[K_1^{(p)}] \quad [Q^{(p)}]^T[K^{'}] \quad [Q^{(p)}]^T[K_2]\right] = \begin{bmatrix} [R_{11}^{(p)}] & [T_1] & [R_{12}^{(p)}] \\ 0 & [T_2] & [R_{22}^{(p)}] \end{bmatrix} \tag{34}$$

where $[T_1]_{dp\times d}$ and $[T_2]_{(N-dp)\times d}$ are extracted from $[Q^{(p)}]^T[K^{'}]=\begin{bmatrix}[T_1]\\ [T_2]\end{bmatrix}$.

We must now triangularize the right term matrix in equation(34). This can be done with a set of Givens rotations. If we decompose only the small sub matrix $[T_2]$, it easily yields:

$$[T_2]=[Q_T]\begin{bmatrix}[R_T]\\ 0\end{bmatrix} \tag{35}$$

where $[R_T]_{d\times d}$ is an upper diagonal matrix and $[Q_T]_{(N-dp)\times(N-dp)}$ is the product of the Givens rotations.

Equation (34) then becomes:

$$[Q^{(p)}]^T[K^{(p+1)}]=\begin{bmatrix}[I_{dpxdp}] & 0\\ 0 & [Q_T]\end{bmatrix}\begin{bmatrix}[R_{11}^{(p)}] & [T_1] & [R_{12}^{(p)}]\\ 0 & [R_T] & \\ 0 & 0 & [Q_T]^T\left[R_{22}^{(p)}\right]\end{bmatrix} \tag{36}$$

$$\begin{bmatrix}[I_{dpxdp}] & 0\\ 0 & [Q_T]^T\end{bmatrix}[Q^{(p)}]^T[K^{(p+1)}]=\begin{bmatrix}[R_{11}^{(p)}] & [T_1] & [R_{12}^{(p)}]\\ 0 & [R_T] & [R_{22}^{'}]\\ 0 & 0 & [R_{22}^{''}]\end{bmatrix} \tag{37}$$

where $[R_{22}^{'}]_{d\times d}$ and $[R_{22}^{''}]_{(N-dp-d)\times d}$ are obtained from multiplication $\begin{bmatrix}[R_{22}^{'}]\\ [R_{22}^{''}]\end{bmatrix}=[Q_T]^T[R_{22}^{p}]$.

It can be seen that the first dp rows of the right hand side in equation (37) are not affected by above transformations and the factor matrix $[R^{(p+1)}]$ at order $p+1$ is thus updated:

$$[R_{11}^{(p+1)}]=\begin{bmatrix}[R_{11}^{(p)}] & [T_1]\\ 0 & [R_T]\end{bmatrix};\ [R_{12}^{(p+1)}]=\begin{bmatrix}[R_{12}^{(p)}]\\ [R_{22}^{'}]\end{bmatrix};\ [R_{22}^{(p+1)}]=[R_{22}^{''}] \tag{38}$$

as well as the Q matrix:

$$[Q^{(p+1)}]=[Q^{(p)}]\begin{bmatrix}[I_{dp\times dp}] & 0\\ 0 & [Q_T]\end{bmatrix} \tag{39}$$

The covariance matrix of the error is also updated:

$$[\hat{E}_{p+1}]=\left[R_{22}^{(p+1)}\right]^T\left[R_{22}^{(p+1)}\right]=\left[R_{22}^{''}\right]^T\left[R_{22}^{''}\right] \tag{40}$$

2.3 Reverse order updating

Consider that, at sample index t, the data matrix $[K^{(p)}]$ of model order p can be partitioned to the data matrix $[K^{(p-1)}]$ by removing its last d columns $[K']_{N\times d}$ of the regressor term:

$$[K^{(p)}]=\begin{bmatrix}\{\varphi(k)\}^T & \{y(k)\}^T\\ \{\varphi(k+1)\}^T & \{y(k+1)\}^T\\ \dots & \dots\\ \{\varphi(k+N-1)\}^T & \{y(k+N-1)\}^T\end{bmatrix}=\left[[K_1^{(p)}]_{N\times dp}\quad [K_2]_{N\times d}\right]=\left[[K_1^{(p-1)}]_{N\times d(p-1)}\quad [K']_{N\times d}\quad [K_2]\right] \quad (41)$$

$$[K^{(p-1)}]=\left[[K_1^{(p-1)}]\quad [K_2]\right] \quad (42)$$

Since the sample number N is always larger than the data dimension d, the data matrix $[K^{(p)}]$ can have the form:

$$[K^{(p)}]=[Q^{(p)}]\begin{bmatrix}[R_{11}^{(p)}] & [R_{12}^{(p)}]\\ 0 & [R_{22}^{(p)}]\end{bmatrix}=[Q^{(p)}]\begin{bmatrix}\underbrace{\begin{bmatrix}[R'_{11}] & [R''_{11}]\\ 0 & [R'''_{11}]\end{bmatrix}}_{R_{11}^{(p)}} & \underbrace{\begin{bmatrix}[R'_{12}]\\ [R''_{12}]\end{bmatrix}}_{[R_{12}^{(p)}]}\\ 0 \quad 0 & [R_{22}^{(p)}]\end{bmatrix} \quad (43)$$

Then we can readily see that:

$$[K^{(p-1)}]=[Q^{(p)}]\begin{bmatrix}[R'_{11}] & [R'_{12}]\\ 0 & [R''_{12}]\\ 0 & [R_{22}^{(p)}]\end{bmatrix}=[Q^{(p)}]\begin{bmatrix}[R'_{11}] & [R'_{12}]\\ 0 & \underbrace{\begin{bmatrix}[R''_{12}]\\ [R_{22}^{(p)}]\end{bmatrix}}_{[R_{22}^{\#}]}\\ 0 & \end{bmatrix}=[Q^{(p)}]\begin{bmatrix}[R'_{11}] & [R'_{12}]\\ 0 & [R_{22}^{\#}]\end{bmatrix}=[Q^{(p)}][R^{\#(p-1)}] \quad (44)$$

and through the exact QR decomposition:

$$[Q^{(p)}][R^{\#(p-1)}]=[Q^{(p-1)}][R^{(p-1)}] \quad (45)$$

It can be seen that matrix $[R^{\#(p-1)}]$ can be found by removing the last d columns from the first sub-columns of matrix $[R^{(p)}]$ and according to (42), it is not an upper triangular matrix. As a result of this, the formulation (44) is therefore not a true QR factorization of the data matrix $[K^{(p-1)}]$. Fortunately, since the first $d(p-1)$ columns of the two matrices $[K^{(p)}]$ and $[K^{(p-1)}]$ are similar, their $[R]$ factors are thus identical in the first $d(p-1)$ rows and $d(p-1)$ columns. That means that the sub-matrices $[R'_{11}]$ and $[R'_{12}]$ in matrix $[R^{\#(p-1)}]$ are exactly as found in the matrix $[R^{(p-1)}]$ to conduct to the updated model parameters at order $p-1$:

$$[\hat{A}^{(p-1)}] = ([R'_{11}]^{-1}[R'_{12}])^T \quad (46)$$

The only component that is different between $[R^{\#(p-1)}]$ and $[R^{(p-1)}]$ lies on the matrix $[R^{\#}_{22}]$ which is not an upper triangular. Note that the energy of matrix $[K^{(p-1)}]$ is unchanged, one can have:

$$\left[Q^{(p)}R^{\#(p-1)}\right]^T\left[Q^{(p)}R^{\#(p-1)}\right] = \left[Q^{(p-1)}R^{(p-1)}\right]^T\left[Q^{(p-1)}R^{(p-1)}\right] \quad (47)$$

Since $[Q^{(p)}]^T[Q^{(p)}] = [Q^{(p-1)}]^T[Q^{(p-1)}] = [I]$, it can be found that:

$$\left[R^{\#(p-1)}\right]^T\left[R^{\#(p-1)}\right] = \left[R^{(p-1)}\right]^T\left[R^{(p-1)}\right] \quad (48)$$

and finally the covariance matrix of error part can be exactly updated:

$$[\hat{E}^{(p-1)}]_{d\times d} = [R^{(p-1)}_{22}]^T[R^{(p-1)}_{22}] = [R^{\#}_{22}]^T[R^{\#}_{22}] = \begin{bmatrix} R^{''T}_{12} & R^{(p)T}_{22} \end{bmatrix}\begin{bmatrix} R^{''}_{12} \\ R^{(p)}_{22} \end{bmatrix} = [R^{''}_{12}]^T[R^{''}_{12}] + [\hat{E}^{(p)}] \quad (49)$$

The QR factorization is accurately updated from model order p to model order $p-1$.

3. NUMERICAL SIMULATIONS

3.1 Computing routine

In operational modal analysis with time varying physical parameters (non stationary systems), it is necessary to identify the variations of modal parameters at each step of computation in a time-frequency scheme. In this paper, since the model is updated with both increased and decreased order, a routine is constructed by combining the three algorithms above to exploit the efficiency of each algorithm in order to progressively searching for an efficient model order and monitoring the change on the modal parameters (Fig.1).

A short sliding window is used on the signal. The routine starts immediately at the beginning of data acquisition with a model at arbitrary order p. This model is updated to the next sample index and then, the order is updated to $p-1$ and $p+1$. An efficient order is chosen within these three order values and the process is continued. It is noted, from above algorithms, that when we combine updating in order and in time, the QR factorization is not the true one but the accurate solution is nevertheless always found.

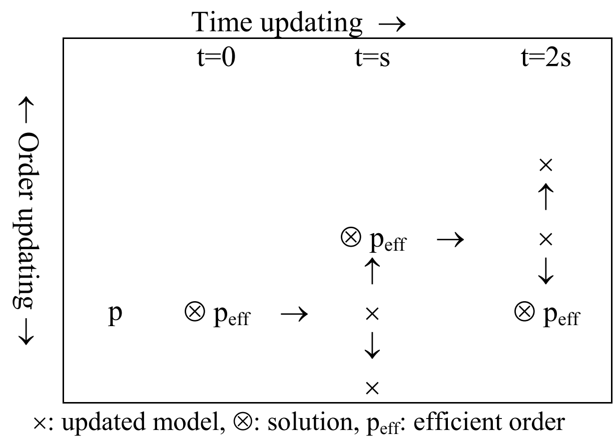


Fig.1 Monitoring routine

3.2 Effect of varying physical parameters and effect of noise

We consider a theoretical system with two degrees of freedom (2 d.o.f.) as shown in Fig.2. Both lumped masses are assumed varying and two cases are investigated: a simultaneously change following a step function as shown in Fig.3-a, and a gradual change following a ramp function as described in Fig.3-b. Responses data are plotted in Fig.4-a and Fig.4-b respectively. A sampling frequency of 200 Hz was applied. Theoretical modal parameters before and after the change are given in Table 1.

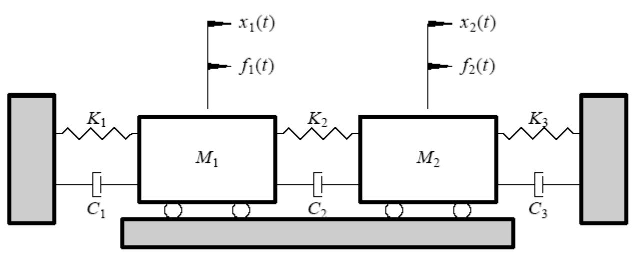
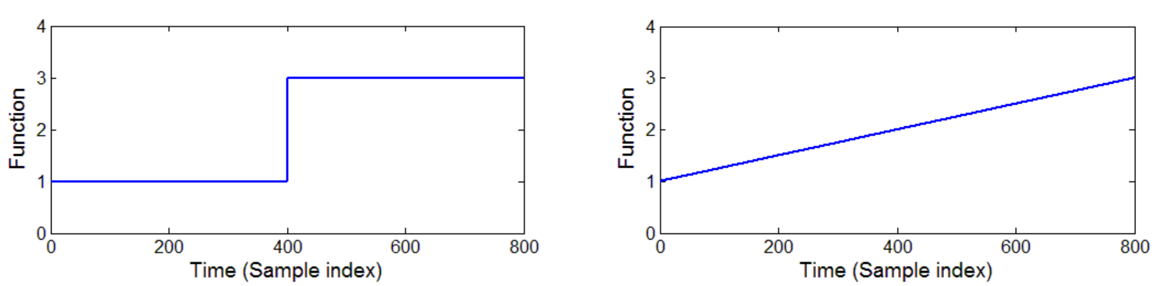
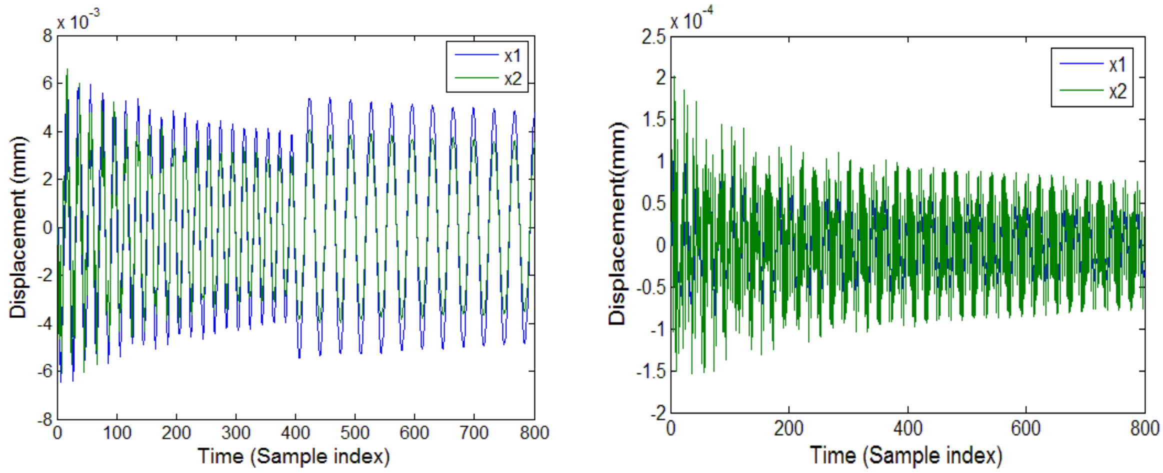


Fig.2 System of 2 degrees of freedom



a. Abrupt change *b. Gradual change at 0.5 times/s*

Fig.3 Masses changing function



a. Abrupt change *b. Gradual change at 0.5 times/s*

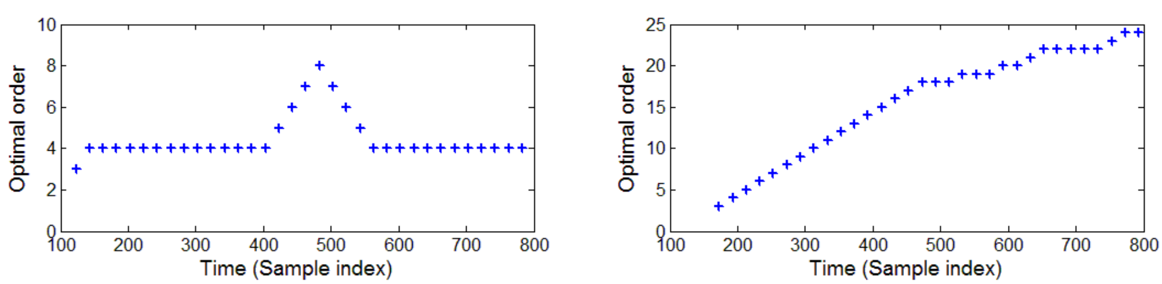
Fig.4 Responses of 2 d.o.f system

Table 1. Modal parameters of 2 d.o.f system

Mode	Before change		After change	
	Frequency (Hz)	Damping rate (%)	Frequency (Hz)	Damping rate (%)
1	10.09	0.28	5.83	0.16
2	37.60	0.72	21.74	0.41

3.2.1 Mass variation with no noisy perturbation

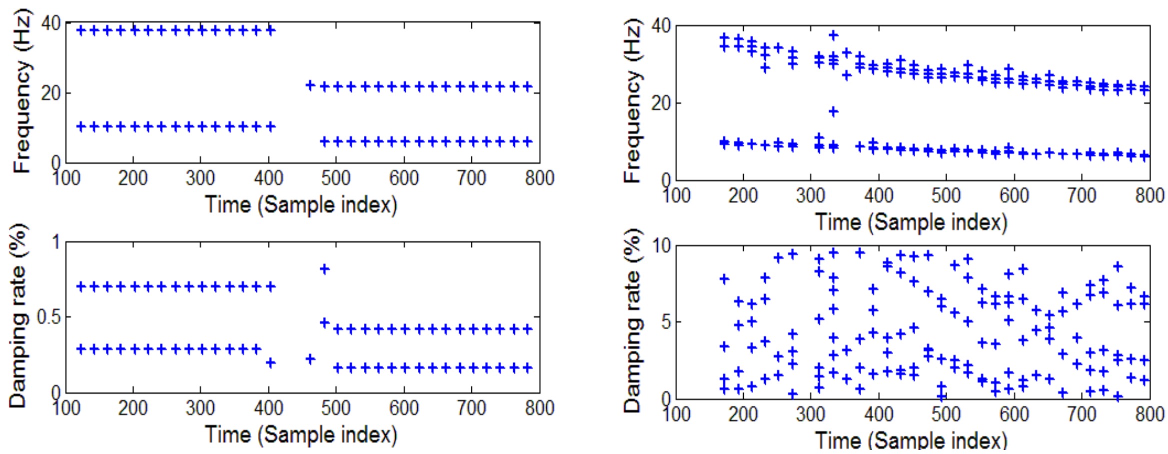
Fig.5 shows the evolution of the order in time. It is seen that the system with constant properties can be monitored with a constant model order except during the transient variation, while if the modal parameters of the system are continuously varying, the order must be continuously adapted.



a. Abrupt change *b. Gradual change at 0.5 times/s*

Fig.5 Monitoring of order

The identified modal parameters are shown in Fig.6. It is found that, with abrupt change (Fig.6-a), the frequency and damping are accurately identified when the masses become again stable. When the masses are continuously varying (Fig.6-b), it can be noticed that the frequency variation is accurately monitored, but not the damping ratios which present a very high variance.

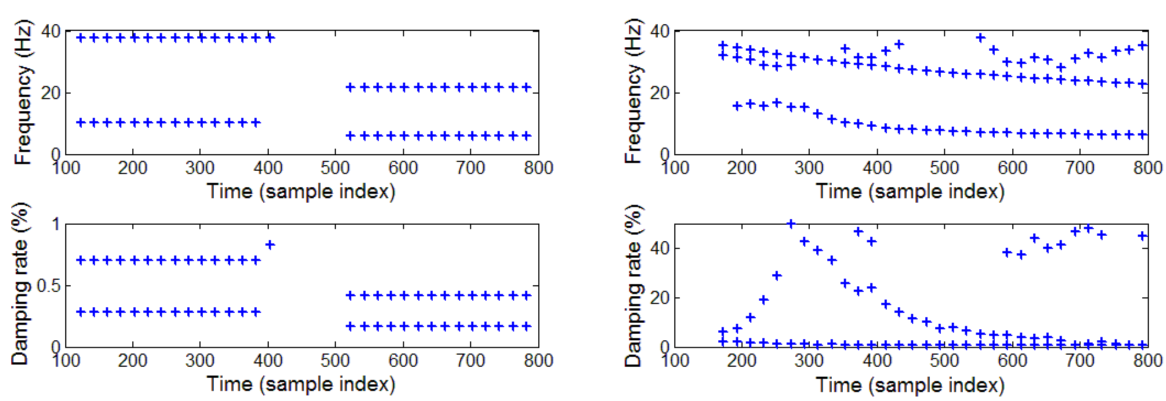


a. Abrupt change *b. Gradual change at 0.5 times/s*

Fig.6 Monitoring of modal parameters (no added noise)

3.2.1 Mass variation with noisy perturbation

Fig.7 shows the monitoring of modal parameters when the data are contaminated by 100% rms (root mean square) random noise. Same observations may be set like the case without noises. The frequency and damping are accurately identified when the masses become again stable (Fig.7-a) while only the frequency can be identified when the physical parameters are continuously varying. Consequently, it may be concluded that a random white noise is not a parameter that affects the accuracy of results.

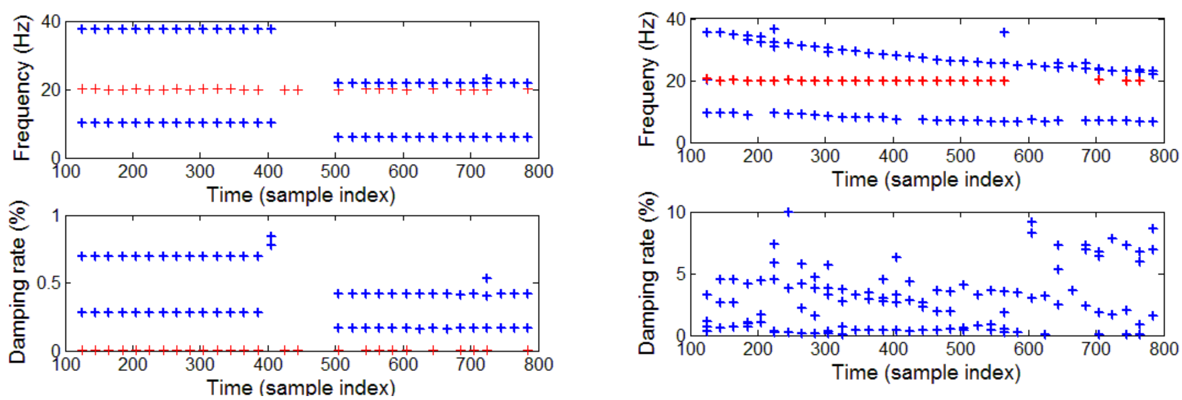


a. Abrupt change *b. Gradual change at 0.5 times/s*

Fig.7 Monitoring of modal parameters with 100% random noise

3.3 Harmonic excitation

The machines are subjected to harmonic excitations and it can be difficult to separate the modal parameters from the excitation frequencies [26]. A sinusoidal excitation at 20Hz has been added to the previous system in both cases reflecting the two types of mass variation. Fig.8-a shows the three frequencies with their variations which are accurately identified in abrupt change case. The identification of the harmonic frequency (20 Hz) is confirmed by a zero-closed damping rate value, even when its frequency becomes closely to a natural frequency (21.7 Hz). On the other hand, Fig.8-b once again confirms that the gradual change deals with a very high variance in identification of the damping ratios; it can even disorder the zero-closed damping rate of the harmonic excitation. Fortunately, the accuracy is still insured on the monitoring of natural frequencies.



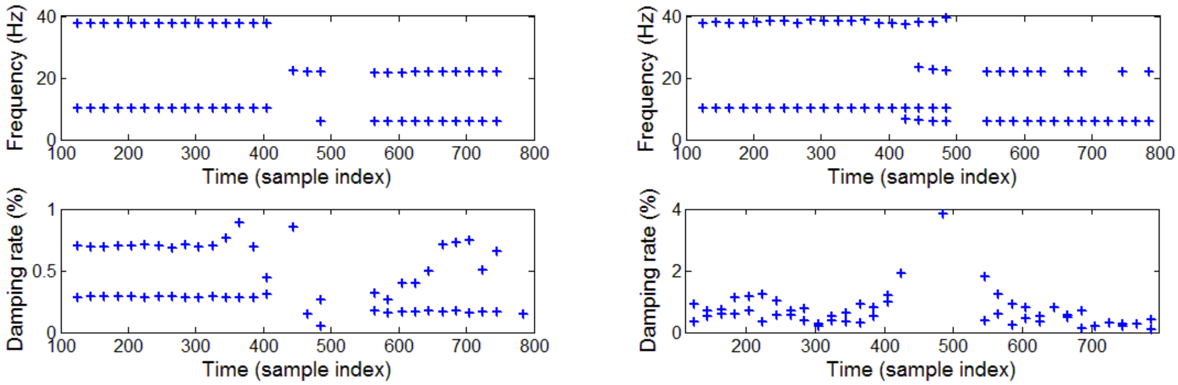
a. Abrupt change *b. Gradual change at 0.5 times/s*

Fig. 8 Monitoring of modal parameters under harmonic excitation

3.4 Random excitation

Taking into account now the random excitation, it is evident that the identification and monitoring of modal parameters changes depend on the randomness of the force hence the variance of excitation is considered at different simulations.

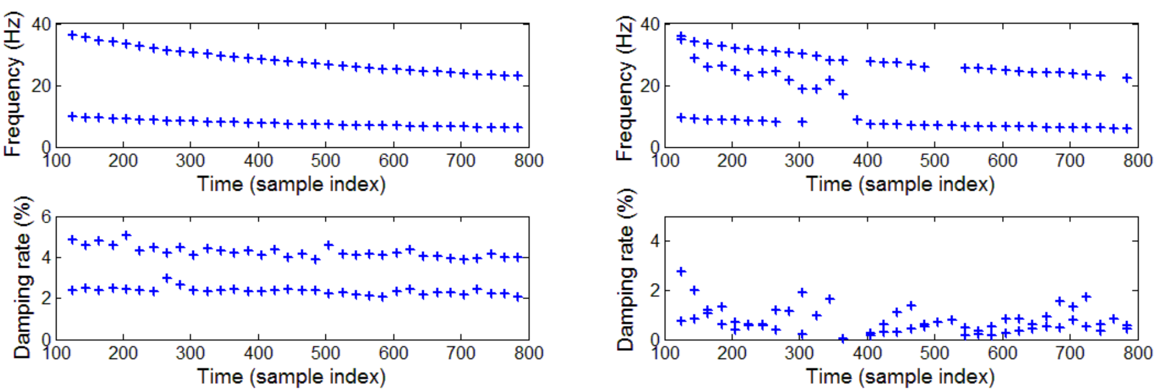
Fig.9 shows the monitoring of modal parameters in the abrupt change case with two different standard derivations (s.t.d.) of the random excitation at 1N and 30N respectively. Various simulations release that both natural frequencies and damping rates can be monitored when the excitation randomness is low. If this latter if high, only frequencies are track-able while the damping ratios are identified with high variances.



a. 1N std random force *b. 30N std random force*

Fig.9 Abrupt change modal parameters under random excitation

The same phenomenon is found in case of gradual change as shown in Fig.10. Natural frequencies are accurately identified and monitored while the damping ratios are dealt with high uncertainty, whatever the randomness of the excitation.



a. 1N std random force *b. 30N std random force*

Fig.10 Gradual change modal parameters under random excitation

3.5 Experimental application

The routine was applied to monitor the modal parameters of a real bridge superstructure where any numerical analysis is available because of an old age of the bridge. The structure was naturally excited by the passing of a heavy truck and the excitation was considered to be random. Three accelerometers were mounted on the middle span to acquire the ambient temporal responses in transversal, vertical and horizontal directions as plotted in Fig.11 at sampling frequency of 200 Hz. As can be seen in Fig.12, the efficient model order used for the fitting of data is changing and is monitored between 4 and 7. Modal parameters are monitored in Fig.13 where first three frequencies were clearly monitored and are more accurate than the short time Fourier at the same configuration (Fig.14). It is seen that when the vehicle moves to the middle span, there is a variation on each frequency within the corresponding frequency ranges of 8 to 6 Hz, 10 to 13 Hz and 25 to 28 Hz respectively. The first mode is the fundamental bending mode and its frequency tends to decrease whereas in the two other frequencies, there is an increasing trend. However the variation of damping rates is cumbersome and according to simulations above, it can be explained by a high randomness of the ambient excitation (Fig.13-b).

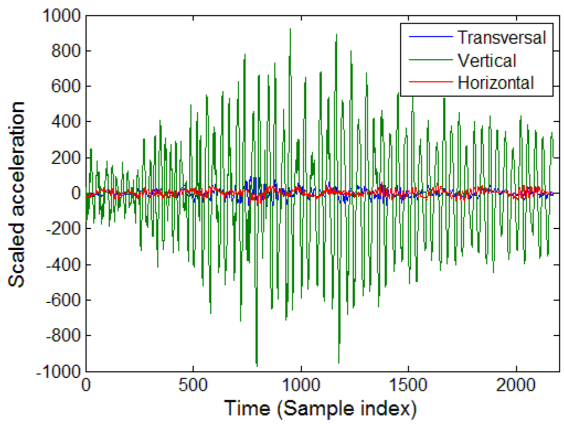


Fig.11 Ambient vibration data

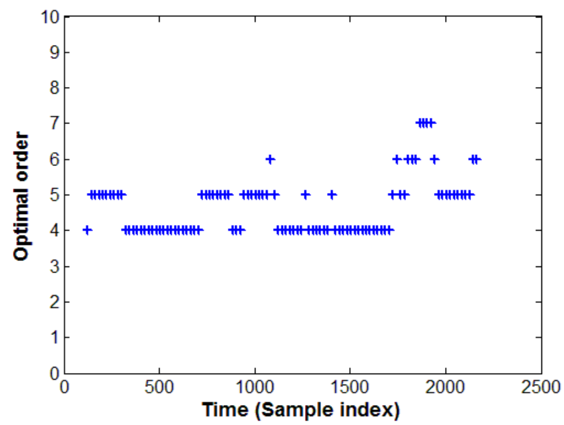
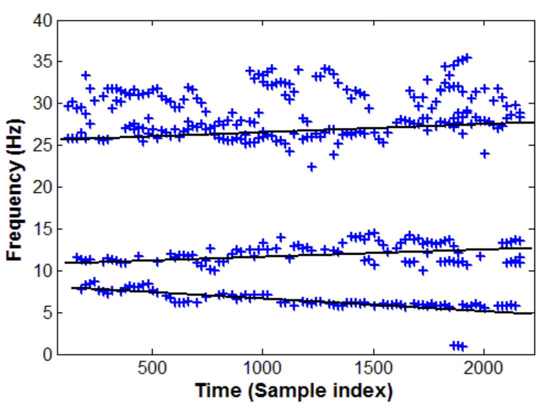
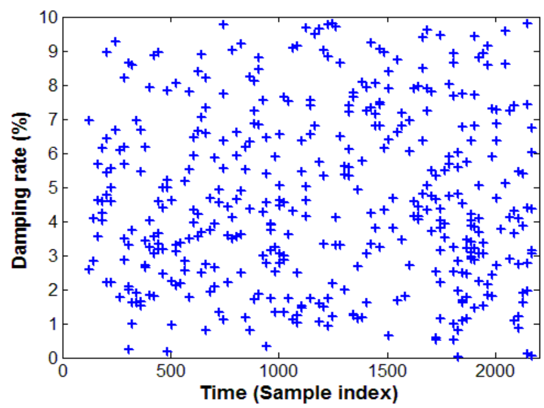


Fig.12 Monitoring of model order



a. Bridge natural frequencies



b. Bridge damping rates

Fig.13 Monitoring of bridge modal parameters

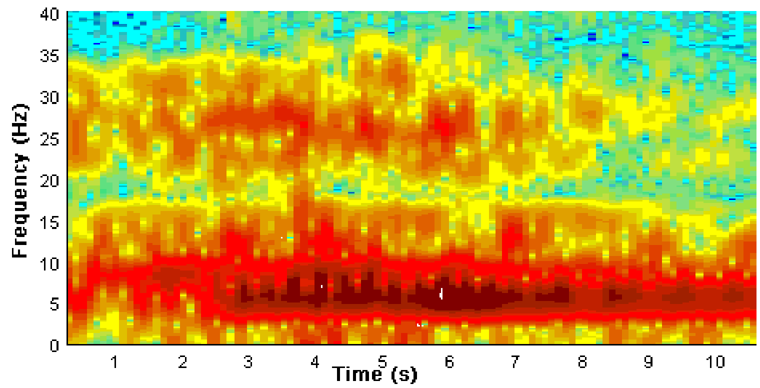


Fig.14 Short time Fourier transform on bridge data

4. CONCLUSION

A method for monitoring modal parameters of systems variations in the time domain has been presented with the using of the multivariate autoregressive short time sliding window modelling. The solution of the least squares method is updated in both the time and model order. With the innovative updating of the QR factorization, only the sub matrices are investigated and the solution is accurately updated when the order either increases or decreases and can be combined with time updating to provide a very fast and effective procedure for monitoring modal parameters. The results from numerical simulations and experimental real applications on a bridge have shown that the proposed method outperforms the short time Fourier transform and can be widely applied to monitor modal parameters variations even following an abrupt or gradual change regardless the white noise. Natural frequencies can be accurately identified and their changes can be well monitored under almost kinds of excitation with various defaults. The monitoring of damping ratios is efficient with the abrupt change which represents a catalectic defect in the system or machine. However, if the default is gradual such as wear in the machine, or if the random excitation is with high standard derivation, the identification of the damping ratios presents a high uncertainty and hence their monitoring is cumbersome.

5. ACKNOWLEDGEMENTS

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7. BIOGRAPHY

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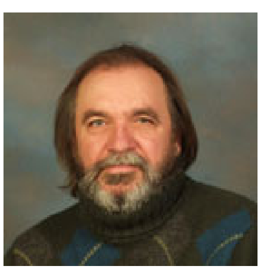

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